A Depletion-Based Reduction of 3D Navier–Stokes Regularity to a Frequency Gain Condition

TalaStar Research Program

1 Introduction

We study the 3D incompressible Navier–Stokes equations:

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \tag{1}$$
$$\nabla \cdot u = 0. \tag{2}$$

The Clay Millennium Problem asks whether smooth divergence-free initial data generate global smooth solutions.

This manuscript reduces global regularity to the verification of a frequency-localized depletion condition.

2 Scaling and Critical Norm

The scaling symmetry:

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$$

A norm is critical if invariant under this scaling.

We choose:

$$\|u\|_{L_t^\infty L_x^3}.$$

3 Dyadic Energy Inequality

Let $u_j = \Delta_j u$.

$$\frac{d}{dt}\|u_j\|_2^2 + c\nu 2^{2j}\|u_j\|_2^2 \le C\|u_j\|_2 \|\Delta_j \mathbb{P}\nabla \cdot (u \otimes u)\|_2. \tag{3}$$

4 Abstract Depletion Hypothesis

There exists $\theta > 0$ such that

$$\|\Delta_j \mathbb{P}\nabla \cdot (u \otimes u)\|_2 \le C\, 2^{j(1-\theta)} B_j(t),$$

with

$$\sum_j 2^{-j} B_j(t) \lesssim \|u(t)\|_{L^3}^2.$$

5 Bootstrap Proposition

Proposition 1. *If the depletion hypothesis holds for all large j, then*

$$\sup_{t\in[0,T]} \|u(t)\|_{L^3} < \infty.$$

Sketch. Insert depletion bound into dyadic inequality. Use Young's inequality to absorb forcing into viscosity term. Sum over high frequencies. Close using Grönwall inequality. □

6 Conditional Regularity

Theorem 1. *If $u \in L^\infty(0, T; L^3)$, then the solution is smooth on $(0, T]$.*

Corollary 1. *Global regularity follows if the depletion hypothesis holds globally.*

7 Conclusion

The Navier–Stokes global regularity problem reduces to proving any mechanism yielding a positive frequency gain $\theta > 0$ in the nonlinear term.