On Frequency-Localized Depletion in the 3D Incompressible Navier–Stokes Equations

TalaStar Research Program

1 Introduction

We study the 3D incompressible Navier–Stokes equations:

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$u(0,x) = u_0(x), \tag{3}$$

where u_0 is smooth, divergence-free, and finite energy.

The goal is to investigate whether frequency-localized nonlinear depletion can produce global a priori bounds in a scaling-critical norm.

2 Scaling and Critical Function Spaces

Definition 1 (Scaling)**.** For $\lambda > 0$, define

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x).$$

Definition 2 (Critical Norm)**.** A norm $\|\cdot\|_X$ is scaling-critical if

$$\|u_\lambda\|_X = \|u\|_X.$$

Examples include:

$$L^\infty_t L^3_x, \quad L^2_t \dot{H}^{1/2}_x, \quad \dot{B}^{-1+3/p}_{p,q}.$$

3 Leray–Hopf Weak Solutions

Theorem 1 (Leray)**.** *For $u_0 \in L^2$, divergence-free, there exists a global weak solution*

$$u \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$$

satisfying the energy inequality:

$$\frac{1}{2}\|u(t)\|^2_{L^2} + \nu \int_0^t \|\nabla u(s)\|^2_{L^2} ds \le \frac{1}{2}\|u_0\|^2_{L^2}.$$

The central open problem is whether such solutions remain smooth for all time.

4 Littlewood–Paley Decomposition

Let $\{\Delta_j\}_{j\in\mathbb{Z}}$ denote dyadic frequency projections.

Definition 3 (Dyadic Blocks)**.**

$$u = \sum_{j\in\mathbb{Z}} \Delta_j u.$$

Apply Δ_j to Navier–Stokes:

$$\partial_t \Delta_j u - \nu \Delta \Delta_j u = -\Delta_j \mathbb{P}\nabla \cdot (u \otimes u). \tag{4}$$

5 Frequency-Localized Energy Inequality

Lemma 1 (Dyadic Energy Inequality)**.** *For each j,*

$$\frac{d}{dt}\|\Delta_j u\|_{L^2}^2 + 2\nu 2^{2j}\|\Delta_j u\|_{L^2}^2 \le C\|\Delta_j u\|_{L^2}\|\Delta_j \mathbb{P}\nabla\cdot(u\otimes u)\|_{L^2}.$$

The nonlinear term governs potential energy transfer across scales.

6 Paraproduct Decomposition

Using Bony decomposition:

$$u \otimes u = T_u u + T_u u + R(u,u).$$

High × High → Low interactions represent the critical obstruction.

7 Proposed Depletion Mechanism

Lemma 2 (Frequency-Localized Depletion Lemma (Conjectural))**.** *There exists $\theta > 0$ such that*

$$\|\Delta_j \mathbb{P}\nabla\cdot(u\otimes u)\|_{L^p} \le C 2^{j(1-\theta)} A_j(t),$$

where $\sum_j 2^{j(-1+3/p)} A_j(t)$ is bounded by a scaling-critical norm of u.

Remark 1. Any $\theta > 0$ yields gain over scale-invariance and would permit closure of a Grönwall-type inequality.

8 Bootstrap to Critical Norm Control

Proposition 1 (Critical Norm Bound)**.** *Assume the Depletion Lemma holds. Then for smooth initial data,*

$$\sup_{t\ge 0}\|u(t)\|_{\dot{B}^{-1+3/p}_{p,q}} \le F(\|u_0\|_{H^m}).$$

9 Global Regularity Conclusion

Theorem 2 (Conditional Regularity)**.** *If $u \in L^\infty_t L^3_x$, then the solution remains smooth.*

Corollary 1. *Under the Depletion Lemma, smooth solutions persist globally.*

10 Discussion and Open Problems

Key unresolved issue: Can one rigorously prove nonlinear depletion sufficient to obtain $\theta > 0$?

Connections to: - Intermittency - Vortex stretching geometry - Alignment phenomena - Anisotropic cascade structure